Some incidence theorems and integrable discrete equations

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Contents

[1] V.E. Adler. Some. . . , *nlin.SI/0409065*.

1 Introduction

An incidence theorem states that geometric objects obtained in different ways coincide. Basically, such incidence may occur in three cases:

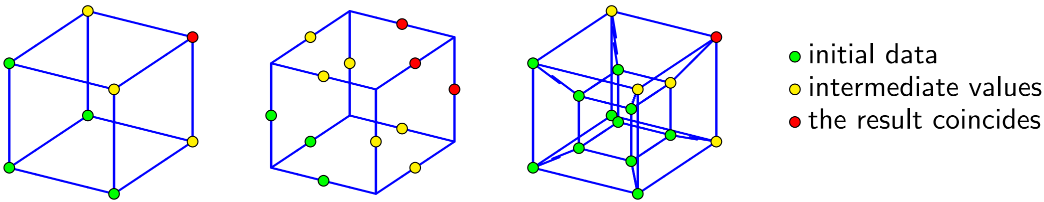
(S) the geometric construction admits some discrete symmetry. For example, "the medians of a triangle are concurrent" ⇔ "pairwise intersections of medians coincide" ⇔ "intersection of two medians is invariant with respect to the action of S_3 on the vertices";

(R) the geometric construction admits a reduction on some submanifold;

(C) the geometric construction can be self-consistently iterated in accordance with some prescribed combinatorics.

Several examples are presented below. Mainly we are interested in (C) with the combinatorics of the cube or hypercube.

Recall, that accordingly to [2, 3], a m-dimensional partial difference equation $F[x] = 0$, $x : \mathbb{Z}^m \to X$, is called $(m+1)D$-consistent if it can be imposed without contradictions on each m-dimensional sublattice in $\mathbb{Z}^{m+1}$. Assume that this equation can be interpreted as a geometric construction which defines some elements of a figure by the other ones. Then the consistency means that some complex figure exists which contains several copies of the basic one.



$3D$-consistency on the vertices; on the edges. $4D$-consistency on the vertices

[2] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* **43A** (2001) 109–123.

[3] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notices* **11** (2002) 573–611.

2 Quadrilateral lattices on the plane

Definition 1. The mapping $x : \mathbb{Z}^3 \to \mathbb{RP}^d$, $d > 2$ is called quadrilateral lattice, if the image of each unit square is a planar quadrilateral.

Theorem 1 ([4]). The mapping x is $4D$-consistent.

The quadrilateral lattice is a very general object which admits numerous reductions: circular lattices, discrete analogues of orthogonal coordinates, discrete asymptotic nets, discrete isothermic surfaces and so on (see e.g. [5, 6] and references therein).

Here we are interested in the case $d = 2$ when Definition 1 becomes senseless. The correct one (that is, preserving the $4D$-consistency) is the following:

Definition 2. The mapping $\mathbb{Z}^3 \to \mathbb{RP}^2$ is called quadrilateral lattice on the plane, if it is a projection of some quadrilateral lattice in space.

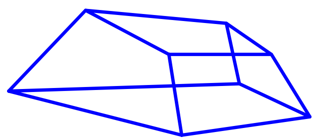
However, this definition is not too constructive. The intrinsic description is based on the following theorem.

[4] A. Doliwa, P.M. Santini. Multidimensional quadrilateral lattices are integrable. *Phys. Lett. A* **233** (1997) 265–372.

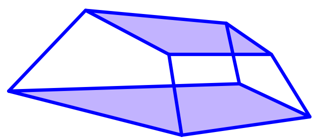
[5] A. Doliwa. Asymptotic lattices and W-congruences in integrable discrete geometry. *J. of Nonl. Math. Phys.* **8** (2001) 88–92.

[6] A.I. Bobenko, Yu.B. Suris. Discrete differential geometry. Consistency as integrability. *math.DG/0504358*.

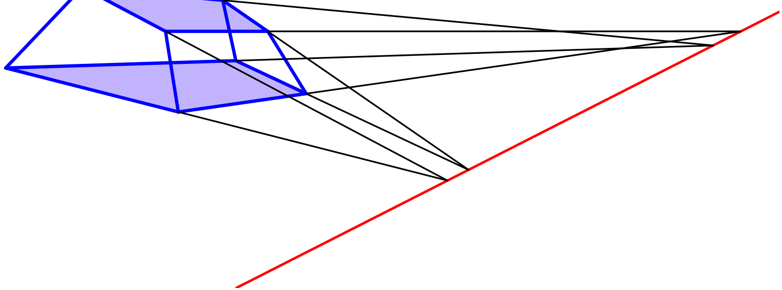
Consider a combinatorial cube on the plane.



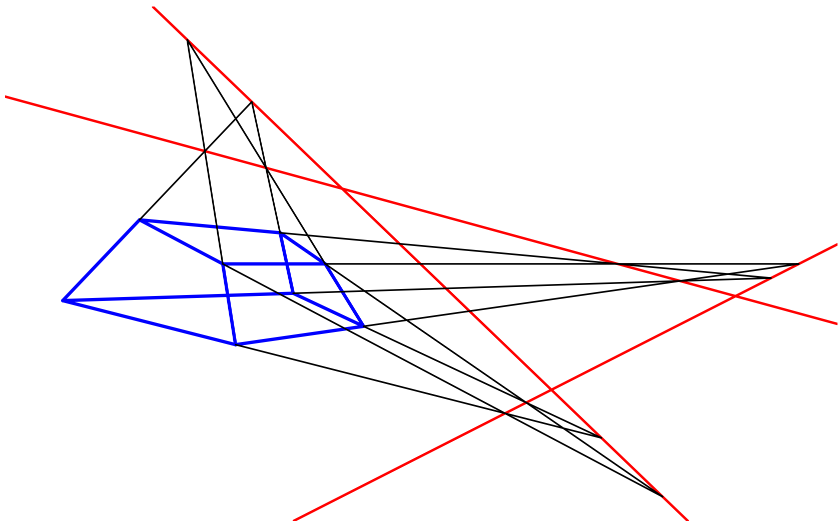
Consider a combinatorial cube on the plane. If, for some pair of the opposite faces,



Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the corresponding edges meet on a straight line,



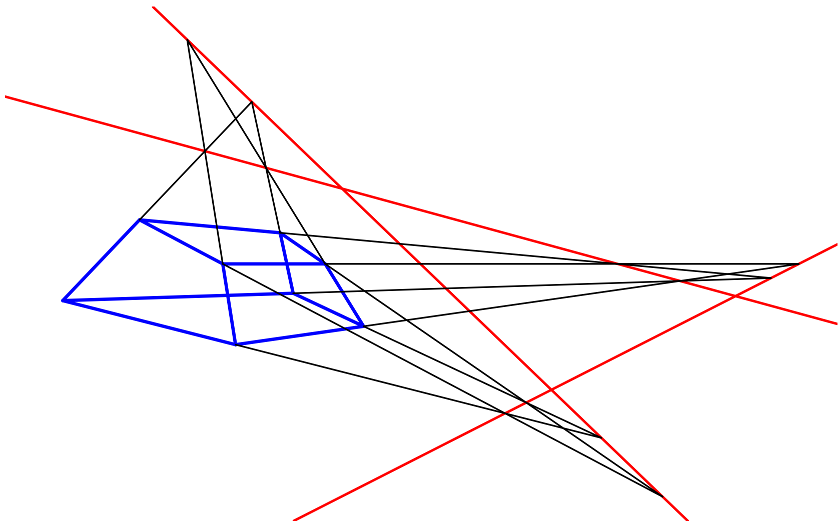
Theorem 2. Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the corresponding edges meet on a straight line, then the same is true for any other pair.



Theorem 2. Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the corresponding edges meet on a straight line, then the same is true for any other pair.

Proof. Collinearity of one quadruple of the intersection points allows to construct a combinatorial cube in space, with planar faces, for which our figure is a projection. For such a figure, edges meet on the intersections of 3 pairs of the planes. □

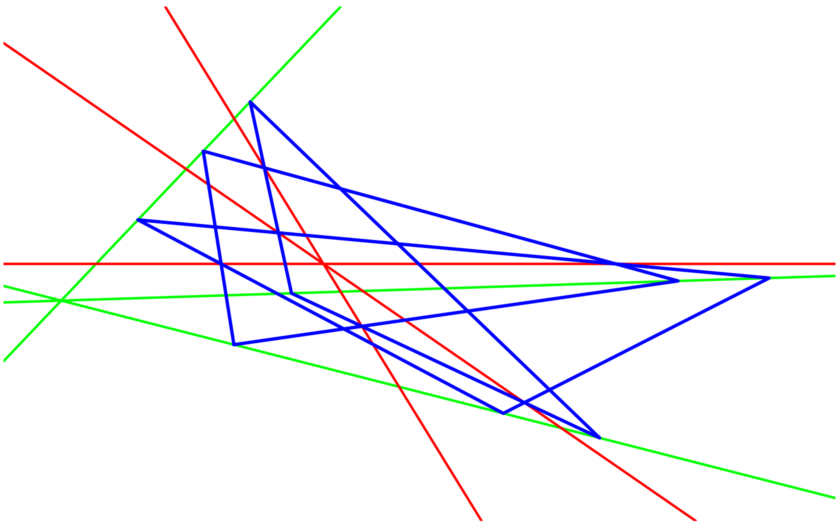
Theorem 2 is of the type (S), with the symmetry group of the cube acting on initial data.



Remark. Actually, the symmetry of the whole figure is even more rich. Namely, 8 vertices of the cube + 12 intersection points and 12 sides + 3 lines of intersections form a regular configuration with the symbol $(20_3 15_4)$. This configuration is mentioned in [7], in connection with the following statement (equivalent to Theorem 2):

> Let 3 triangles be perspective with the common center. Then 3 axes of perspective of 3 pairs of triangles meet in one point.

[7] F. Levi, Geometrische Konfigurationen, Leipzig: 1929, pp. 143, 202.



Definition 3. The mapping $\mathbb{Z}^3 \to \mathbb{RP}^2$ is called quadrilateral lattice on the plane, if the image of any unit cube is the figure described in the Theorem 2, that is, the images of the corresponding edges of any pair of the opposite faces meet on a straight line.

Collinearity of 4 intersection points is the condition, which allows to construct any vertex of the combinatorial cube by the other ones. This defines the mapping $(\mathbb{RP}^2)^7 \to \mathbb{RP}^2$. Let $X, X_1, \ldots, X_{23}$ be given, then X_{123} is defined by

$$\begin{gathered} A = XX_1 \cap X_3X_{13}, \quad B = XX_2 \cap X_3X_{23} \\ A' = X_2X_{12} \cap AB, \quad B' = X_1X_{12} \cap AB \\ X_{123} = A'X_{23} \cap B'X_{13}. \end{gathered} \tag{1}$$

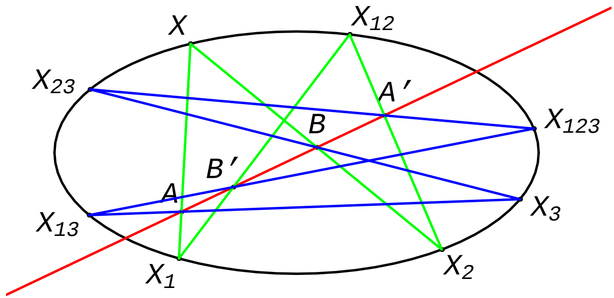
Theorem 2 means that the result is invariant with respect to the permutations of the subscripts.

Corollary 3. The mapping (1) is $4D$-consistent.

Proof. This follows from the Theorem 1 and the proof of the Theorem 2. □

3 Double cross-ratio equation

Quadrilateral lattice on the plane admits the reduction to a conic section C, that is, if the points $X, X_1, \dots, X_{23}$ lie on C then the point X_{123} defined by eqs (1) lies on C as well.



The rational parameters x of the points X on the conic satisfy the double cross-ratio equation, or discrete Schwarz-BKP [8]

$$\frac{(x-x_{12})(x_{13}-x_{23})}{(x_{12}-x_{13})(x_{23}-x)} = \frac{(x_{123}-x_3)(x_2-x_1)}{(x_3-x_2)(x_1-x_{123})}.$$

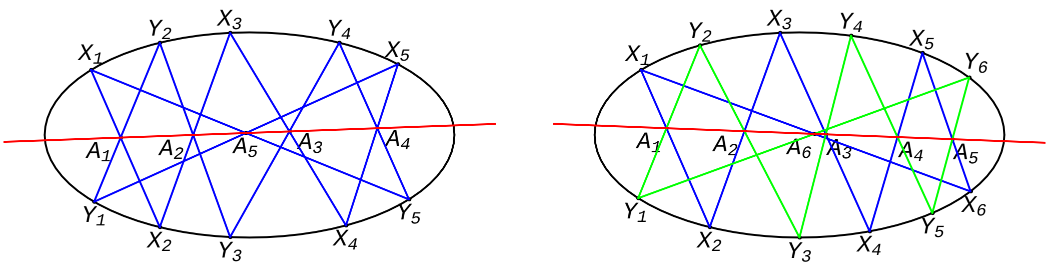
[8] B.G. Konopelchenko, W.K. Schief, Reciprocal figures, graphical statics and inversive geometry of the Schwarzian BKP hierarchy. *Stud. Appl. Math.* **109:2** (2002) 89–124.

This reduction is a particular case of Möbius theorem on inscribed polygons.

Theorem 4 (Möbius). *Let $X_1, Y_1, \ldots, X_N, Y_N$ be points on a conic. Consider the intersection points $A_j = X_jX_{j+1} \cap Y_jY_{j+1}$, $j = 1, \ldots N-1$ and*

$$A_N = \begin{cases} X_NY_1 \cap Y_NX_1 & \text{if} \quad N = 2n+1, \\ X_NX_1 \cap Y_NY_1 & \text{if} \quad N = 2n. \end{cases}$$

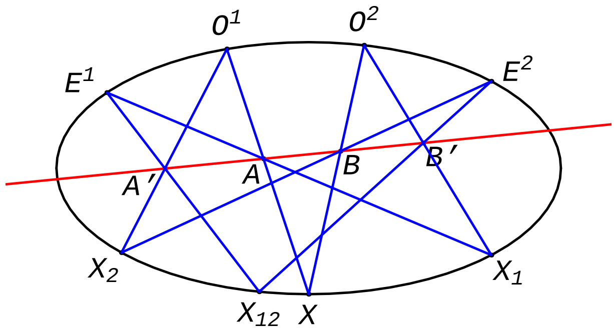
If all of these points except possibly one are collinear then the same is true for the remaining point.



[9] F.A. Möbius, Verallgemeinerung des Pascal'schen Theorems das in einen Kegelschnit beschriebene Sechseck betreffend. *J. Reine Angew. Math.* **36** (1848) 216–220.

4 Hietarinta equation

Consider one more, very similar, analog of Pascal theorem.



Theorem 5. Let X, X_1, X_2 and O^1, O^2, E^1, E^2 be points on a conic C. Then the point X_{12} defined as follows lies on C as well.

$$
\begin{aligned}
A &= XO^1 \cap X_1E^1, \quad B = XO^2 \cap X_2E^2 \\
A' &= X_2O^1 \cap AB, \quad B' = X_1O^2 \cap AB \\
X_{12} &= E^1A' \cap E^2B'.
\end{aligned}
\tag{2}
$$

The corresponding values of the rational parameter on C are related by equation

$$(x, e^2, x_1, o^1, x_{12}, o^2, x_2, e^1) = 1 \tag{3}$$

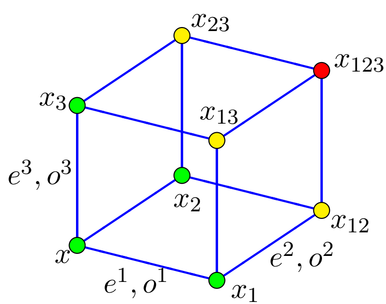
introduced by Hietarinta in the recent paper [10]. Here brackets denote multi-ratio $(a, b, c, d, \dots) = (a-b)/(b-c) \cdot (c-d)/\dots$. (The ordering of its arguments is obtained from the ordering in octagon $XO^1X_2E^2X_{12}E^1X_1O^2$ by skips over two vertices.)

The values e^i, o^i are parameters of the equation, associated to the edges of the square lattice.

Theorem 6. Equation (3) is $3D$-consistent, that is, the mapping $x : \mathbb{Z}^3 \to \mathbb{CP}^1$ governed by equation

$$(x, e^j, x_i, o^i, x_{ij}, o^j, x_j, e^i) = 1$$

for any 2-dimensional sublattice is correctly defined.

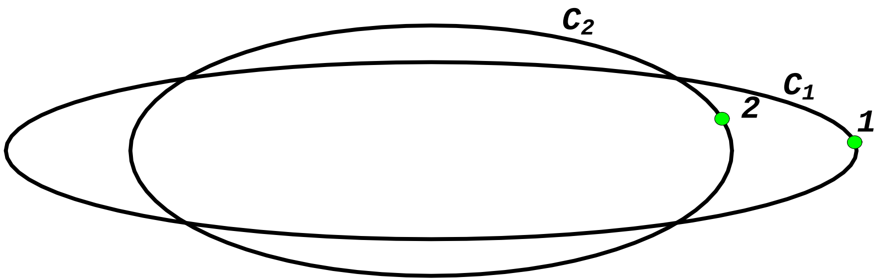


[10] J. Hietarinta. A new two-dimensional lattice model that is "consistent around a cube". *J. Phys. A* **37:6** (2004) L67–73.

5 Yang-Baxter mappings on the linear pencils of conics

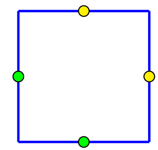
Our last example is $3D$-consistency on the edges of a cube.

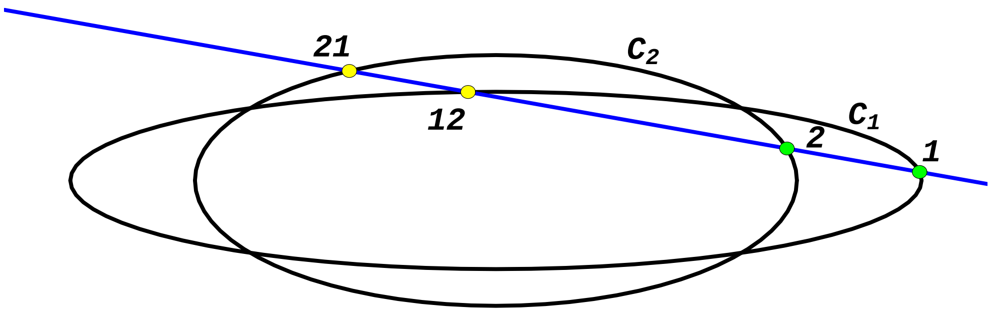
Let X_1, X_2 be points on the conic sections C_1, C_2 respectively.



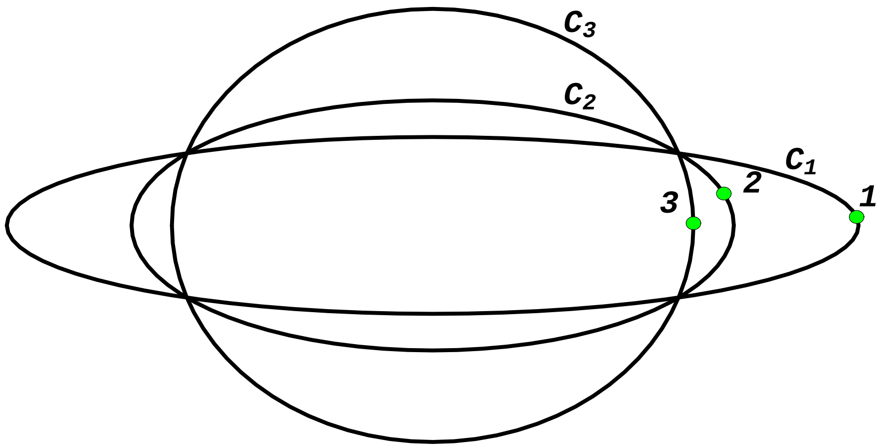
The mapping $F_{12} : C_1 \times C_2 \to C_1 \times C_2$ is defined as follows:

$$X_{12} = X_1 X_2 \cap C_1, \quad X_{21} = X_1 X_2 \cap C_2$$



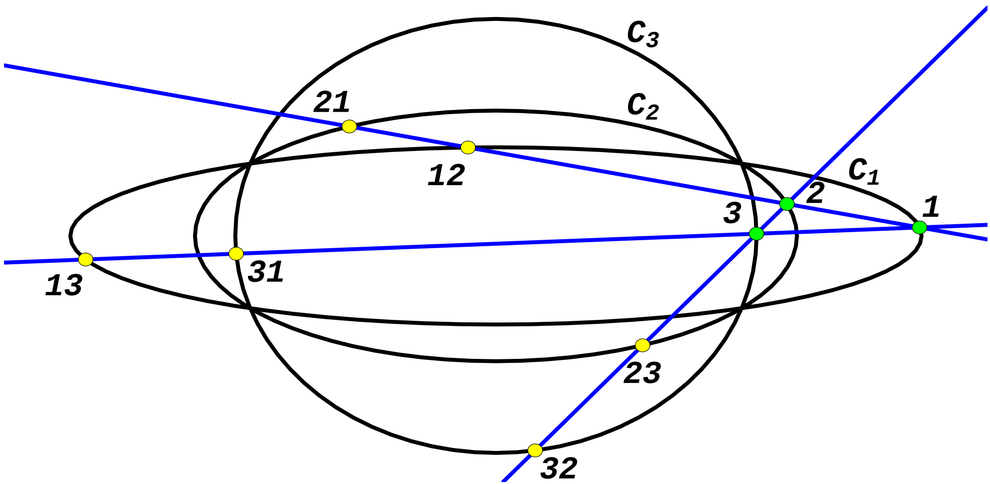


Consider the initial data on three conics from the linear pencil.



Consider the initial data on three conics from the linear pencil.

Apply the mappings $F_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

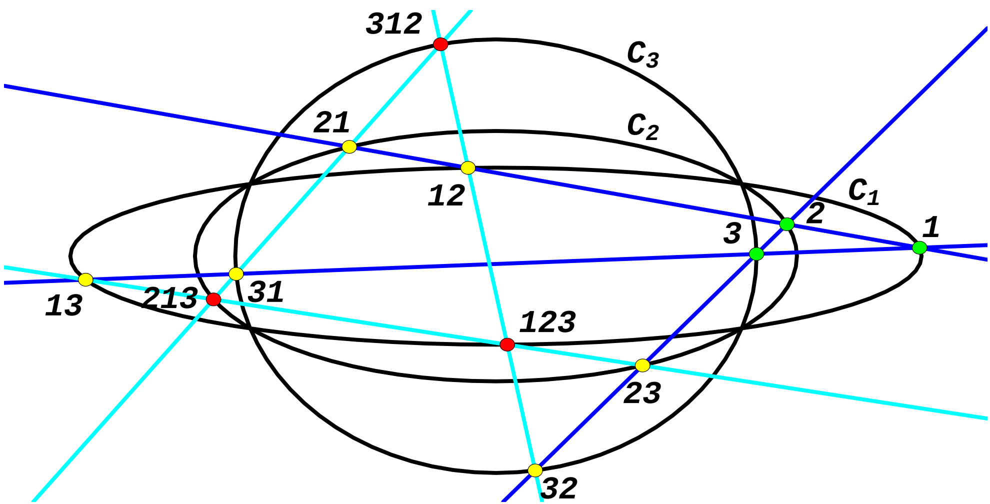


Consider the initial data on three conics from the linear pencil.

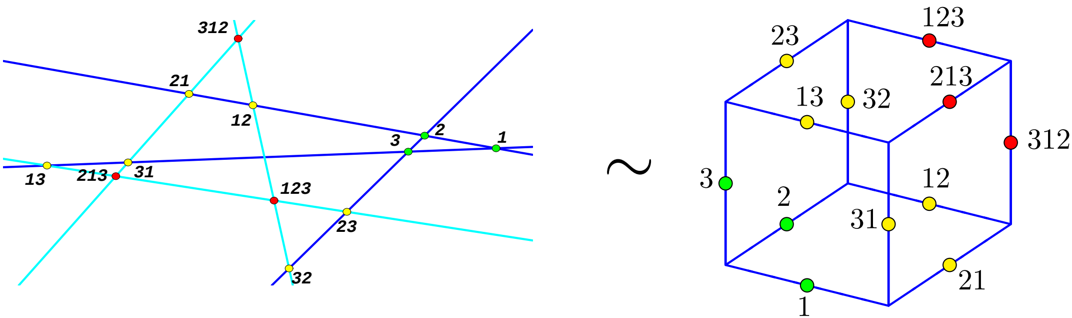
Apply the mappings $F_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

Apply the mappings once more. Let $F_{ij} : (X_{ik}, X_{jk}) \mapsto (X_{ikj}, X_{jki})$.

Theorem 7. The mappings F_{ij} are $3D$-consistent: $X_{ijk} = X_{ikj}$.



12 points and 6 lines can be identified with the edges and the faces of a cube. $3D$-consistency with this combinatorics is actually equivalent to the notion of Yang-Baxter maps [11, 12].



[11] V.G. Drinfeld. On some unsolved problems in quantum group theory. *Lect. Notes in Math.* **1510** (1992) 1–8.

[12] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett A* **314** (2003) 214–221.

Under a rational parametrization of the conics $C_i : X_i = X_i(x_i)$ the mapping F_{12} turns into a birational mapping on $\mathbb{CP}^1 \times \mathbb{CP}^1$. There exist 5 projective types of the linear pencils of conics $C_i = C + a_i K$ [13]. These types lead to the following list of the mappings [14] $(i, j \in \{1, 2\})$:

$$x_{ij} = a_i x_j \frac{(1 - a_2)x_1 + a_2 - a_1 + (a_1 - 1)x_2}{a_2(1 - a_1)x_1 + (a_1 - a_2)x_2 x_1 + a_1(a_2 - 1)x_2}$$

$$x_{ij} = \frac{x_j}{a_i} \cdot \frac{a_1 x_1 - a_2 x_2 + a_2 - a_1}{x_1 - x_2}$$

$$x_{ij} = \frac{x_j}{a_i} \cdot \frac{a_1 x_1 - a_2 x_2}{x_1 - x_2}$$

$$x_{ij} = x_j \left(1 + \frac{a_2 - a_1}{x_1 - x_2}\right)$$

$$x_{ij} = x_j + \frac{a_1 - a_2}{x_1 - x_2}$$

The first one corresponds to the above figures with 4-point locus.

[13] M. Berger. Geometry. *Springer-Verlag, Berlin* 1987.
[14] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. *Comm. Anal. and Geom.* **12:5** (2004) 967–1007.